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IMPROVEMENT OF K_2-STABILITY UNDER TRANSITIVE ACTIONS OF ELEMENTARY GROUPS

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For any ring A with finite stable rank $\mathrm{sr}(A)$ the canonical map $K_2(n-1,A)\to K_2(A)$ is known to be surjective, if $n\geq \mathrm{sr}(A)+2$ and bijective, if $n\geq \mathrm{sr}(A)+3$ (cf. [3, 4, 7, 9, 11]). Moreover, examples are known where these bounds are sharp (cf. [4]). Thus we have to impose additional assumptions on A to get an improvement of the stability behaviour of K_2. We show that the bounds improve by 1, provided certain actions of elementary groups are transitive. As a consequence we get a result, proved by van der Kallen [5] with a different technique, that the map $K_2(2,A)\to K_2(A)$ is surjective and the map $K_2(3,A)\to K_2(A)$ is bijective, if A is a Dedekind ring of arithmetic type as in Bass–Milnor–Serre [2] and with infinitely many units.

I am indebted to W. van der Kallen, whose very helpfully remarks lead to the final version of this paper.

Let A be any ring with 1 and let $n\geq 3$. Given $z\in A$, let $G_z(n-1,A)$ denote the subgroup of the general linear group $\mathrm{GL}(n-1,A)$ consisting of all matrices with last row equal to $(z\gamma, 1+z\delta)$, $\gamma\in A^{n-2}$, $\delta\in A$, and let $E_z(n-1,A)$ denote the subgroup of the elementary group $E(n-1,A)$, generated by all elementary matrices from $G_z(n-1,A)$. For each $z,y\in A$, the group $E_{zy}(n-1,A)$ acts via right multiplication on the group $G_{zy}(n-1,A)\cap E_z(n-1,A)$. We say that this action is transitive, if $G_{zy}(n-1,A)\cap E_z(n-1,A)$ is contained in $\mathrm{GL}(n-2,A)\cdot E_{zy}(n-1,A)$. For each non-zero $z\in A$, let $\mathrm{T}(n-1,z)$ denote the following condition:

$\mathrm{T}(n-1,z)$: There is a two-sided ideal I_z, such that $\mathrm{sr}(A/I_z)=1$, and for each y which is a unit mod I_z the group $E_{zy}(n-1,A)$ acts transitively on $G_{zy}(n-1,A)\cap E_z(n-1,A)$.

Another condition, we have to impose on A, is van der Kallen's 2-fold stable range condition SR^2_{n+1} (cf. [5]), which replaces Bass' stable range condition SR_{n+1}. Recall that A satisfies SR^2_{n+1}, if for any two unimodular rows $(a_1,\dots,a_{n+1})$,

$(b_1,\dots,b_{n+1})$ there exist $c_1,\dots,c_n$, such that $(a_1+a_{n+1}c_1,\dots,a_n+a_{n+1}c_n)$ and $(b_1+b_{n+1}c_1,\dots,b_n+b_{n+1}c_n)$ are again unimodular. Let $\mathrm{sr}^2(A)$ be the minimal n (or ∞), such that A satisfies SR^2_{n+1}. For example, if A is finitely generated over a central subring R whose maximal spectrum is noetherian of Krull dimension $d\geq 1$, we have $\mathrm{sr}^2(A)\leq d+1$ (cf. [5]).

To state our main result, let $S(n-1,A)$ denote the inverse image of $E(n,A)\cap \mathrm{GL}(n-1,A)$ in the Steinberg group $\mathrm{St}(n,A)$.

Theorem. *Let A be a ring with finite 2-fold stable rank $\mathrm{sr}^2(A)\geq 2$, let $n\geq \mathrm{sr}^2(A)+1$ and assume that $E(n-1,A)$ acts transitively on unimodular rows of length $n-1$.*

(a) *If* $\mathrm{T}(n-1,1)$ *holds, we have* $S(n-1,A)=\mathrm{im}(\mathrm{St}(n-1,A))$, *and hence $K_2(n-1,A)\to K_2(A)$ is surjective and $K_1(n-1,A)\to K_1(A)$ is an isomorphism.*

(b) *If* $\mathrm{T}(n-1,z)$ *holds for each non-zero z, the map* $\mathrm{St}(n,A)\to\mathrm{St}(n+1,A)$ *is injective, hence $K_2(n,A)\to K_2(A)$ is an isomorphism.*

Corollary (Van der Kallen [5]). *Let A be a Dedekind ring of arithmetic type and with infinitely many units. Then the map $K_2(2,A)\to K_2(A)$ is surjective and the map $K_2(3,A)\to K_2(A)$ is an isomorphism.*

Proof of the corollary. Since A is 1-dimensional, we have $\mathrm{sr}^2(A)=2$. To go further we use the following deep arithmetic results of Vaserstein [10], corrected and extended by Liehl [8]:

(1) $\mathrm{SL}(2,A)=E(2,A)$.

(2) For each $z\in A$, $E_z(2,A)$ is a normal subgroup of $G_z(2,A)\cap E(2,A)$,

(3) $G_z(2,A)\cap E(2,A)/E_z(2,A)$ is a finite cyclic group of order $r(z)$, where $r(z)$ depends only on the $\mathfrak{p}$-adic valuation of z, for primes $\mathfrak{p}$ which divide the order m of the group of roots of unity of A.

(4) Let $(zy,1+z\delta)$ be the last row of an element σ from $G_z(2,A)\cap E(2,A)$. Then $\sigma\in E_z(2,A)$ if and only if the $r(z)$-th power norm residue symbol $(\frac{1+z\delta}{zy})_{r(z)}$ is trivial.

Since any unimodular row of length 2 can be completed to an element of $\mathrm{SL}(2,A)$, (1) implies that $E(2,A)$ acts transitively on unimodular rows of length 2. For each non-zero $z\in A$ let $I_z:=I$ be the product of all primes dividing m. Thus A/I is semi-local. If y is a unit mod I, we get from (3) that $r(zy)=r(z)$ and thus (4) implies that the canonical map from $G_{zy}(2,A)\cap E(2,A)/E_{zy}(2,A)$ to $G_z(2,A)\cap E(2,A)/E_z(2,A)$ is an isomorphism. Hence $G_{zy}(2,A)\cap E_z(2,A)=E_{zy}(2,A)$ for all y which are units mod I, hence $\mathrm{T}(2,z)$ holds for all $z\neq 0$.

The method, which leads to a proof of the theorem, is adopted from [7], where the prestability problem for K_2 was solved. We briefly recall the relevant notations: The subspace of A^n of all vectors, whose kth component is zero, is denoted by A^n_k. Given $q\in A^n_k$, let $C_k(q):=\prod_{i\neq k}x_{ik}(q_i)$ and $R_k(q):=\prod_{i\neq k}x_{ki}(q_i)$. These elements from the Steinberg group $\mathrm{St}(n,A)$ should be viewed as a kth 'column' resp. 'row'. Consequently we view q as a column (resp. row) vector, if it occurs as $C_k(q)$ (resp.

$R_k(q)$). Thus for any $n\times n$-matrix B the notations $C_k(B\cdot q)$ and $R_k(q\cdot B)$ make sense.

For $z\in A$ a z-pair (x^z, x_z) in $\mathrm{St}(n,A)$ consists of two elements x^z, x_z from $\mathrm{St}(n,A)$, which have presentations

$$x^z=\varrho\prod_{i=1}^{m}(R_n(a_i)\cdot C_n(b_iz)),$$

$$x_z=\varrho\prod_{i=1}^{m}(R_n(za_i)\cdot C_n(b_i)),$$

with $a_i, b_i\in A_n^n$ for $i=1,\dots,m$ and $\varrho\in S(n-1,A)$.

Let $f_n:\mathrm{St}(n,A)\to E(n,A)$ denote the canonical projection. The group $W(n,A)$ is defined to be the normal subgroup of $\mathrm{St}(n,A)$ generated by the following elements:

(i) $tR_n(a)t^{-1}R_n(-a\cdot f_n(t)^{-1})$, $tC_n(a)t^{-1}C_n(-f_n(t)\cdot a)$ with $t\in S(n-1,A)$,

(ii) $x^z(x_z)^{-1}$, (x^z,x_z) a z-pair with $x^z, x_z\in S(n-1,A)$,

(iii) $C_n(c)R_n(b)C_n(cz)R_n(-b)C_n(-c)R_n(-zb)$, where $b,c\in A_n^n$ satisfy $b\cdot c^t=-1$.

This group $W(n,A)$ solves the prestability problem for K_2 (cf. [7]). Thus to prove part (b) of the theorem, we have to show that $W(n,A)$ is trivial. Actually the theorem is true under a slightly weaker transitivity assumption, which we define now. For each non-zero $z\in A$ let $\mathrm{T}'(n-1,z)$ denote the following condition:

$T'(n-1,z)$: Given an arbitrary unimodular vector $b=(b_1,\dots,b_{n-1})$ from A^{n-1} the group $E_{zy}(n-1,A)$ acts transitively on $G_{zy}(n-1,A)\cap E_z(n-1,A)$ for some $y:=b_{n-1}+\sum_{i=1}^{n-2}b_iu_i$.

Obviously, $\mathrm{T}(n-1,z)$ implies $\mathrm{T}'(n-1,z)$. The proof of the following lemma will be postponed til the end of the paper.

Lemma 1. *Let $z\in A$, $z\neq 0$ and*

$$M^z=C_n(az)R_n(b)C_n(cz)R_n(d)C_n(-c'z)R_n(-b')C_n(-a'z),$$

$$M_z=C_n(a)R_n(zb)C_n(c)R_n(zd)C_n(-c')R_n(-zb')C_n(-a')$$

a z-pair, such that $M^z, M_z\in S(n-1,A)$ and b or b' are unimodular. If $E(n-1,A)$ acts transitively on unimodular rows of length $n-1$ and if $\mathrm{T}'(n-1,z)$ holds, we have $M^z=M_z\in\mathrm{im}(\mathrm{St}(n-1,A)$.

Proof of the theorem. Let V_n denote the set of elements x of $\mathrm{St}(n,A)$ which have a presentation $x=\varrho C_n(a)R_n(b)C_n(c)R_n(d)$ with $\varrho\in\mathrm{im}(\mathrm{St}(n-1,A))$. A *$z$-pair from V_n* is a z-pair (x^z,x_z) from $\mathrm{St}(n,A)$ such that

$$x^z=\varrho C_n(az)R_n(b)C_n(cz)R_n(d),$$

$$x_z=\varrho C_n(a)R_n(zb)C_n(c)R_n(zd),$$

with $\varrho\in\mathrm{im}(\mathrm{St}(n-1,A))$.

Lemma 2. *Let (x^z, x_z) be a z-pair from V_n. Let $t^z := x^z C_n(qz)$, $t_z := x_z C_n(q)$. Then (t^z, t_z) is again a z-pair from V_n.*

Proof. Let

$$x^z = \varrho C_n(az) R_n(b) C_n(cz) R_n(d),$$

$$x_z = \varrho C_n(a) R_n(zb) C_n(c) R_n(zd),$$

with $\varrho \in \mathrm{im}(\mathrm{St}(n-1, A))$. Look at the last row $(\gamma, 1+\delta z)$ of $f_n(t^z)$ and at the last row $(d, 1+dq^t z)$ of $R_n(d)C_n(qz)$. Since SR_n^2 holds, we find $d' \in A^{n-1}$, such that $b' := \gamma - (1+\delta z)d'$ and $p := d - (1+dq^t z)d'$ are both unimodular. Hence there are $c', s \in A^{n-1}$, such that $b'c'^t = \delta$ and $ps^t = dq^t$. Thus for suitably chosen $a', \upsilon \in A^{n-1}$ the following four elements lie in $S(n-1, A)$:

$$M^z := t^z R_n(-d') C_n(-c'z) R_n(-b') C_n(-a'z),$$

$$M_z := t_z R_n(-zd') C_n(-c') R_n(-zb') C_n(-a'),$$

$$N^z := R_n(d) C_n(qz) R_n(-d') C_n(-sz) R_n(-p) C_n(-\upsilon z),$$

$$N_z := R_n(zd) C_n(q) R_n(-zd') C_n(-s) R_n(-zp) C_n(-\upsilon).$$

Since p is unimodular, Lemma 1 implies that $N^z = N_z =: \tau$ lies in the image of $\mathrm{St}(n-1, A)$. Now we can write

$$R_n(d) C_n(qz) R_n(-d') = \tau C_n(\upsilon z) R_n(p) C_n(sz)$$

and

$$R_n(zd) C_n(q) R_n(-zd') = \tau C_n(\upsilon) R_n(zp) C_n(s).$$

We insert these expressions into M^z and M_z and get

$$M^z = \varrho C_n(az) R_n(b) C_n(cz) \tau C_n(\upsilon z) R_n(p) C_n((s-c')z) R_n(-b') C_n(-a'z),$$

$$M_z = \varrho C_n(a) R_n(zb) C_n(c) \tau C_n(\upsilon) R_n(zp) C_n(s-c') R_n(-zb') C_n(-a').$$

Since τ lies in the image of $\mathrm{St}(n-1, A)$, we can move it to the right using the Steinberg relations. Now we apply again Lemma 1 to M^z and M_z and get $M^z = M_z =: \sigma \in \mathrm{im}(\mathrm{St}(n-1, A))$ and thus

$$t^z = \sigma C_n(a'z) R_n(b') C_n(c'z) R_n(d'),$$

$$t_z = \sigma C_n(a') R_n(zb') C_n(c') R_n(zd'),$$

as claimed. Note that the vector b' is unimodular.

First of all we apply Lemma 2 with $z=1$. The result shows that V_n is stable under multiplication from the right by elements from $\mathrm{St}(n, A)$, hence $V_n = \mathrm{St}(n, A)$. Moreover, the proof showed that any $x \in V_n$ has a presentation $x = \varrho C_n(a) R_n(b) C_n(c) R_n(d)$ with b unimodular. Thus, if $x \in S(n-1, A)$, Lemma 1 implies that x lies in the image of $\mathrm{St}(n-1,A)$, hence we are done with part (a) of the theorem.

To prove that the group $W(n,A)$ is trivial, first note, that generators of type (i) vanish, since $S(n-1,A)=\mathrm{im}(\mathrm{St}(n-1,A))$. Lemma 2 implies that any z-pair from $\mathrm{St}(n,A)$ actually is a z-pair from V_n and again Lemma 1 shows that generators of type (ii) vanish. Thus we are left with generators of type (iii). Let $M:=C_n(c)R_n(b)C_n(cz)R_n(-b)C_n(-c)R_n(-zb)$ with $z\in A$ and $b,c\in A_n^n$, such that $bc^t=-1$. Since b is unimodular and $E(n-1,A)$ acts transitively on unimodular rows, we find $\sigma\in\mathrm{im}(\mathrm{St}(n-1,A))$, such that $\sigma R_n(b)\sigma^{-1}=x_{n1}(1)$. Thus

$$\sigma M\sigma^{-1}=C_n(c')x_{n1}(1)C_n(c'z)x_{n1}(-1)C_n(-c')x_{n1}(-z)$$

with $c_1'=-1$. Hence, if we let as usual for a unit u

$$w_{ij}(u):=x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u),$$

we get

$$\begin{aligned}\sigma M\sigma^{-1}&=\prod_{i\neq 1}x_{in}(c_i')w_{1n}(-1)C_n(c'z)w_{1n}(1)\prod_{i\neq 1}x_{in}(-c_i')x_{n1}(-z)\\&=\prod_{i\neq 1}x_{in}(c_i')\prod_{i\neq n}x_{i1}(-c_i'z)x_{n1}(z)\prod_{i\neq 1}x_{in}(-c_i')x_{n1}(-z)\\&=1.\end{aligned}$$

Proof of Lemma 1. Without loss of generality we may assume, that b' is unimodular. First note that the statement of the lemma is unchanged, if we replace M^z,M_z by $\varrho M^z,\varrho M_z$ or by $M^z\varrho,M_z\varrho$ with $\varrho\in\mathrm{im}(\mathrm{St}(n-1,A))$. Thus, since $E(n-1,A)$ acts transitively on unimodular rows, we can arrange that $b_{n-1}'=0$. Now, if we move successively $x_{n-1,n}(-c_{n-1}'z)$ (resp. $x_{n-1,n}(-c_{n-1}')$), $x_{n,n-1}(d_{n-1})$ (resp. $x_{n,n-1}(zd_{n-1})$) and $x_{n-1,n}(c_{n-1}z)$ (resp. $x_{n-1,n}(c_{n-1})$) to the right, we see that without loss of generality we may assume that $c_{n-1}=d_{n-1}=0$ and b' is unimodular. Since $\mathrm{T}'(n-1,z)$ holds, there is a vector $u\in A_{n-1}^{n-1}$ such that $E_{zy'}(n-1,A)$ acts transitively on $G_{zy'}(n-1,A)\cap E_z(n-1,A)$, where $y':=b_{n-1}'+\sum_{i=1}^{n-2}b_i'u_i$. If we multiply M^z,M_z from the right by $C_{n-1}(u)$, which lies in the image of $\mathrm{St}(n-1,A)$, we can replace b_{n-1}' by y' without changing the condition $c_{n-1}=d_{n-1}=0$. Clearly then with $y:=-y'$, the group $E_{zy}(n-1,A)$ acts transitively on $G_{zy}(n-1,A)\cap E_z(n-1,A)$ as well. Hence we finally have reduced to the special case that M^z,M_z satisfy $c_{n-1}=d_{n-1}=0$ and $E_{zy}(n-1,A)$ acts transitively on $G_{zy}(n-1,A)\cap E_z(n-1,A)$, where $y=-b_{n-1}'$. Given $t\in A_n^n$, we let $\hat{t}$ denote the vector obtained from t in setting the $n-1$-th component equal to 0. Define

$$S^z:=R_{n-1}(\hat{b})C_{n-1}(cz)R_{n-1}(d)C_{n-1}(-\hat{c}'z)R_{n-1}(-\hat{b}'),$$

$$S_z:=R_{n-1}(z\hat{b})C_{n-1}(c)R_{n-1}(zd)C_{n-1}(-\hat{c}')R_{n-1}(-z\hat{b}').$$

Clearly (S^z,S_z) is a z-pair from the image of $\mathrm{St}(n-1,A)$. An easy calculation now shows that

$$f_n(M^z)=f_n(M_z)=\begin{pmatrix}\alpha & \beta^t zy & 0\\ \gamma & 1+\delta zy & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$f_n(S^z)=\begin{pmatrix}\alpha & \beta^t z & 0\\ y\gamma & 1+y\delta z & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$f_n(S_z)=\begin{pmatrix}\alpha & \beta^t & 0\\ zy\gamma & 1+zy\delta & 0\\ 0 & 0 & 1\end{pmatrix},$$

where α is an $(n-2)\times(n-2)$-matrix, $\beta,\gamma\in A^{n-2}$, $\delta\in A$. Thus, especially, $f_n(S_z)\in G_{zy}(n-1,A)\cap E_z(n-1,A)$, hence the assumptions imply that we find a zy-pair (w^{zy},w_{zy}) from the image of $\mathrm{St}(n-1,A)$ such that

$$f_n(M_z\cdot w^{zy})=f_n(M^z\cdot w^{zy})=f_n(S_z\cdot w_{zy})=f_n(S^z\cdot w_y^z)\in \mathrm{GL}(n-2,A).$$

Now Lemma 2.4 of [7] implies that $M_z\cdot w^{zy}=S_z\cdot w_{zy}$ and $M^z\cdot w^{zy}=S^z\cdot w_y^z$, hence M^z and M_z lie in the image of $\mathrm{St}(n-1,A)$. To see that $M^z=M_z$, note that $(S^z\cdot w_y^z, S_z\cdot w_{zy})$ is a z-pair from the image of $\mathrm{St}(n-1,A)$, hence $(S^z\cdot w_y^z)\cdot(S_z\cdot w_{zy})^{-1}$ lies in the image of $W(n-1,A)$, hence is trivial by Theorem 1.2 of [7]. Clearly this implies $M^z=M_z$.

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